

t CG TORSION PAIRS

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ABSTRACT. We investigate conditions for when the t -structure of Happel-Reiten-Smalø associated to a torsion pair is a compactly generated t -structure. The concept of a t CG torsion pair is introduced and for any ring R , we prove that $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is a t CG torsion pair in $R\text{-Mod}$ if, and only if, there exists, $\{T_\lambda\}$ a set of finitely presented R -modules in \mathcal{T} , such that $\mathcal{F} = \bigcap \text{Ker}(\text{Hom}_R(T_\lambda, ?))$. We also show that every t CG torsion pair is of finite type, and study when these two classes of torsion pair coincide for a given coherent ring.

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INTRODUCTION

The notion of a *torsion pair* was introduced in the sixties by Dickson (see [Dic66]) in the setting of abelian categories, generalizing the classical notions for abelian groups. Since then many are the applications that torsion pairs have within the study of localizations, tilting theory, categories theory, etc. Indeed, the equivalent version in the setting of triangulated category is the concept of t -structure introduced by Beilinson, Bernstein and Deligne [BBD82], in their study of the perverse sheaves over an analytic or algebraic variety stratified by some closed subset. This notion allows us to associate, to an object of an arbitrary triangulated category, its corresponding “objects of homology”, which belong to some abelian subcategory of such triangulated category. Such subcategory is called the heart of the t -structure.

In the nineties, Happel, Reiten and Smalø observed that there is a natural way to associate a t -structure to the derived category of a given abelian category endowed with a torsion pair, even when such derived category is not a category in the strict sense of categories (see [HRS96]). This t -structure is perhaps the most well known t -structure for triangulated categories. Nevertheless, other t -structures, such as the compactly generated t -structure have been well documented in the literature

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also (see [ATJLS10], [ATJLSS03], [PSar]). For some of these compactly generated t -structures, and when certain conditions on the ambient triangulated category are imposed, an explicit description is obtained for the rightmost class of the t -structure, or co-aisle (see [ATJLSS03]). For these two types of t -structures, several authors have investigated conditions for when the heart of such t -structures is a Grothendieck category or a module category (see [CGM07], [CMT11], [HKM02], [MT12], [PS15], [PS16a], [PS16b], [PSar]).

In particular, [PSar] shows that over a commutative Noetherian ring R , the heart of almost every compactly generated t -structure in $\mathcal{D}(R)$, the derived category of the ring R , is a Grothendieck category. On the other hand, Theorem 3.7 in [PS15] shows that a countable direct limits of exact sequences in the heart of a compactly generated t -structure is exact. Recall that over a Grothendieck category direct limits of exact sequences are exact.

Hence the following question seems natural to ask: is the heart of a compactly generated t -structure in $\mathcal{D}(R)$ an AB5 category? Asked in such a general way, this question is unapproachable. Therefore, we tackle this question for the t -structure of Happel-Reiten-Smalø. The main goal of this article is to provide a positive answer to this question, through the concept of tCG torsion pairs (see Definition 2.1), and to study the relation of torsion pairs of finite type and the tCG torsion pairs, for a certain classes of coherent rings.

The organization of this paper is as follows. In Section 1 we give all the preliminaries and terminology needed in the rest of the paper. Section 2 introduces the reader to the notion of a tCG torsion pair, and shows Theorem 2.3, which gives a characterization result for the tCG torsion pairs. This characterization result, then allows us to further characterize the tCG torsion pair for Noetherian rings (see Theorem 2.9) and also to establish, for a coherent ring R , an injective function between the set of the torsion pairs in $fp(R\text{-Mod})$ and the set of the tCG torsion pairs (see Theorem 2.11). Finally, in Section 3, we prove a result that relates the tCG torsion pair and a TTF triple in $R\text{-Mod}$, assuming that R is a von Neumann regular ring; we also show in this section a way of constructing tCG torsion pairs from torsion pairs of finite type.

1. PRELIMINARES AND TERMINOLOGY

The concepts that we shall introduce in this section are applied in the case of module categories, but sometimes we will use them in the more general context of Grothendieck categories and it is in this context that we define them. In the sequel, \mathcal{G} will denote a Grothendieck category. For a class of objects \mathcal{S} in \mathcal{G} , we will use the following notation $\mathcal{S}^\perp := \{X \in \mathcal{G} \mid \text{Hom}_{\mathcal{G}}(S, X) = 0, \text{ for all } S \in \mathcal{S}\}$ and ${}^\perp\mathcal{S} := \{X \in \mathcal{G} \mid \text{Hom}_{\mathcal{G}}(X, S) = 0, \text{ for all } S \in \mathcal{S}\}$.

A *torsion pair* in \mathcal{G} is a pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ of full subcategories of \mathcal{G} satisfying the following two conditions:

- (1) $\mathcal{T} = {}^\perp \mathcal{F}$ and $\mathcal{F} = \mathcal{T}^\perp$;
- (2) For each object X of \mathcal{G} , there is an exact sequence

$$0 \longrightarrow T_X \longrightarrow X \longrightarrow F_X \longrightarrow 0$$

where $T_X \in \mathcal{T}$ and $F_X \in \mathcal{F}$. In such case the objects T_X and F_X are uniquely determined, up to isomorphism, and the assignment $X \rightsquigarrow T_X$ (resp. $X \rightsquigarrow F_X$)

underlies a functor $t : \mathcal{G} \rightarrow \mathcal{T}$ (resp. $(1 : t) : \mathcal{G} \rightarrow \mathcal{F}$), which is right (resp. left) adjoint to the inclusion functor $\mathcal{T} \hookrightarrow \mathcal{G}$ (resp. $\mathcal{F} \hookrightarrow \mathcal{G}$).

The composition $\mathcal{G} \xrightarrow{t} \mathcal{T} \hookrightarrow \mathcal{G}$ (resp. $\mathcal{G} \xrightarrow{(1:t)} \mathcal{F} \hookrightarrow \mathcal{G}$), which we will still denote by t (resp. $(1 : t)$), is called the *torsion radical* (resp. *torsion coradical*) associated to \mathbf{t} . The torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is called *hereditary* when \mathcal{T} is closed under taking subobjects in \mathcal{G} .

Let X and V be objects of \mathcal{G} . We say that X is *V-generated* (resp. *V-presented*) when there is an epimorphism $V^{(I)} \twoheadrightarrow X$ (resp. an exact sequence $V^{(J)} \rightarrow V^{(I)} \rightarrow X \rightarrow 0$), for some set I (resp. sets I and J). We will denote by $\text{Gen}(V)$ and $\text{Pres}(V)$ the classes of V -generated and V -presented objects, respectively. An object V will be called *1-tilting object* when $\text{Gen}(V) = \text{Ker}(\text{Ext}_R^1(V, ?))$. If V is a 1-tilting object, we have that $\text{Gen}(V) = \text{Pres}(V)$ and that the pair $(\text{Gen}(V), \text{Ker}(\text{Hom}_R(V, ?)))$ is a torsion pair in \mathcal{G} which is called *the tilting torsion pair associated to V*. A *classical 1-tilting object* is a 1-tilting object V such that the canonical morphism $\text{Hom}_{\mathcal{G}}(V, V)^{(I)} \rightarrow \text{Hom}_{\mathcal{G}}(V, V^{(I)})$ is an isomorphism, for all sets I . By [CDT97, Proposition 1.2], we know that if $\mathcal{G} = R\text{-Mod}$, then a classical 1-tilting R -module is just a finitely presented 1-tilting R -module.

We refer the reader to [Nee01] for the precise definition of *triangulated category*, but diverting from the terminology in that book, for a given triangulated category \mathcal{D} , we will denote by $?[1] : \mathcal{D} \rightarrow \mathcal{D}$ its suspension functor. We will then make $?[0] = 1_{\mathcal{D}}$, and $?[k]$ will denote the k -th power of $?[1]$, for each integer k . The (distinguished) triangles in \mathcal{D} will be denoted by $X \rightarrow Y \rightarrow Z \xrightarrow{+}$. Recall that if \mathcal{D} and \mathcal{A} are a triangulated and an abelian category, respectively, then an additive functor $H : \mathcal{D} \rightarrow \mathcal{A}$ is a *cohomological functor* when, given any triangle $X \rightarrow Y \rightarrow Z \xrightarrow{+}$, one gets the following induced long exact sequence in \mathcal{A} :

$$\cdots \longrightarrow H^{n-1}(Z) \longrightarrow H^n(X) \longrightarrow H^n(Y) \longrightarrow H^n(Z) \longrightarrow \cdots$$

where $H^n(?) := H \circ (?[n])$, for each integer n .

Given a triangulated category \mathcal{D} , a *t-structure* in \mathcal{D} is a pair $(\mathcal{U}, \mathcal{W})$ of full subcategories, closed under taking direct summands in \mathcal{D} , which satisfy the following assertions:

- (1) $\text{Hom}_{\mathcal{D}}(U, W[-1]) = 0$, for all $U \in \mathcal{U}$ and for all $W \in \mathcal{W}$;
- (2) $\mathcal{U}[1] \subseteq \mathcal{U}$;
- (3) For each $X \in \text{Ob}(\mathcal{D})$, there is a triangle $U \rightarrow X \rightarrow V \xrightarrow{+}$ in \mathcal{D} , where $U \in \mathcal{U}$ and $V \in \mathcal{W}[-1]$.

It is easy to see that in such case $\mathcal{W} = \mathcal{U}^{\perp}[1]$ and $\mathcal{U} = {}^{\perp}(\mathcal{W}[-1]) = {}^{\perp}(\mathcal{U}^{\perp})$. For this reason, we will write a *t-structure* as $(\mathcal{U}, \mathcal{U}^{\perp}[1])$. We will call \mathcal{U} and \mathcal{U}^{\perp} the *aisle* and the *co-aisle* of the *t-structure*, respectively. The objects U and V above triangle are uniquely determined by X , up to isomorphism, and define functors $\tau_{\mathcal{U}} : \mathcal{D} \rightarrow \mathcal{U}$ and $\tau^{\mathcal{U}^{\perp}} : \mathcal{D} \rightarrow \mathcal{U}^{\perp}$ which are right and left adjoints to the respective inclusion functors, and we call such functors the *left* and *right truncation functors* with respect to the given *t-structure*. The full subcategory $\mathcal{H} = \mathcal{U} \cap \mathcal{W} = \mathcal{U} \cap \mathcal{U}^{\perp}[1]$ is called the *heart* of the *t-structure* and it is an abelian category, where the short exact sequences are the triangles in \mathcal{D} with their three terms in \mathcal{H} . Moreover, with the obvious abuse of notation, the assignments $X \rightsquigarrow (\tau_{\mathcal{U}} \circ \tau^{\mathcal{U}^{\perp}[1]})(X)$ and $X \rightsquigarrow (\tau^{\mathcal{U}^{\perp}[1]} \circ \tau_{\mathcal{U}})(X)$ define naturally isomorphic functors $\mathcal{D} \rightarrow \mathcal{H}$ which are cohomological (see [BBD82]).

If $\mathcal{D} = \mathcal{D}(R)$ and \mathcal{S} is a set of objects in \mathcal{D} , the smallest full subcategory of \mathcal{D} closed under coproducts, extensions and positive shifts is the aisle of a t -structure (cf. [ATJLSS03, Proposition 3.2]). In that case, if $(\mathcal{U}, \mathcal{U}^\perp[1])$ denoted such t -structure, then \mathcal{U}^\perp consists of the $Y \in \mathcal{D}$ such that $\text{Hom}_{\mathcal{D}}(S[n], Y) = 0$, for all $S \in \mathcal{S}$ and for all integers $n \geq 0$. In this case, we will write that $\mathcal{U} = \text{aisle}(\mathcal{S})$, moreover, the t -structure $(\text{aisle}(\mathcal{S}), \text{aisle}(\mathcal{S})^\perp[1])$ is called *compactly generated* if \mathcal{S} consist of compact objects (i.e. for each $S \in \mathcal{S}$ the functor $\text{Hom}_{\mathcal{D}}(S, ?)$ commute with coproducts) and we say that \mathcal{S} is a *set of compact generators* of the aisle. On the other hand, the compact objects of $\mathcal{D}(R)$ are the complexes which are quasi-isomorphic to bounded complexes of finitely generated projective modules (see [Ric89]).

For the rest of this section, we assume that R is a commutative Noetherian ring and we denote by $\text{Spec}(R)$ its spectrum. A subset Z of $\text{Spec}(R)$ is *stable under specialization* if, for any pair of prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$, with $\mathfrak{p} \in Z$, it holds that $\mathfrak{q} \in Z$. Equivalently, Z is a union of closed subsets of $\text{Spec}(R)$, with respect Zariski topology of $\text{Spec}(R)$. Such a subset will be called *sp-subset* in the sequel. The typical example is the support of an R -module N , denoted $\text{Supp}(N)$, which consists of the prime ideals \mathfrak{p} such that $N_{\mathfrak{p}} := R_{\mathfrak{p}} \otimes_R N \neq 0$. We have the following precise description for hereditary torsion pairs in $R\text{-Mod}$ (see [Ste75, Chapter VII]).

Proposition 1.1. *The assignment $Z \rightsquigarrow (\mathcal{T}_Z, \mathcal{T}_Z^\perp)$ defines a bijection between the sp-subsets of $\text{Spec}(R)$ and the hereditary torsion pairs in $R\text{-Mod}$, where \mathcal{T}_Z is the class of the R -modules T such that $\text{Supp}(T) \subseteq Z$. Its inverse takes $(\mathcal{T}, \mathcal{T}^\perp)$ to the set $Z_{\mathcal{T}}$ of prime ideal \mathfrak{p} such that R/\mathfrak{p} is in \mathcal{T} .*

A *filtration by supports* of $\text{Spec}(R)$ is a decreasing map $\phi : \mathbb{Z} \rightarrow \text{P}(\text{Spec}(R))$, such that $\phi(i)$ is an sp-subset for each $i \in \mathbb{Z}$. We will refer to a filtration by supports of $\text{Spec}(R)$ simply by an sp-filtration of $\text{Spec}(R)$. In [ATJLS10], Alonso, Jeremías and Saorín associated to each sp-filtration $\phi : \mathbb{Z} \rightarrow \text{P}(\text{Spec}(R))$, the t -structure $(\mathcal{U}_\phi, \mathcal{U}_\phi^\perp[1])$ in $\mathcal{D}(R)$, where $\mathcal{U}_\phi := \text{aisle}(\{R/\mathfrak{p}[-i] : i \in \mathbb{Z} \text{ and } \mathfrak{p} \in \phi(i)\})$.

2. t CG TORSION PAIRS

Through out this section, R is a unital associative ring and $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ is a torsion pair in $R\text{-Mod}$. The t -structure of *Happel-Reiten-Smalø* in $\mathcal{D}(R)$ associated to the torsion pair \mathfrak{t} , is the given by $(\mathcal{U}_{\mathfrak{t}}, \mathcal{U}_{\mathfrak{t}}^\perp[1]) := (\mathcal{U}_{\mathfrak{t}}, \mathcal{W}_{\mathfrak{t}})$, where:

$$\mathcal{U}_{\mathfrak{t}} := \{X \in \mathcal{D}^{\leq 0}(R) \mid H^0(X) \in \mathcal{T}\} \quad \text{and} \quad \mathcal{W}_{\mathfrak{t}} := \{X \in \mathcal{D}^{\geq -1}(R) \mid H^{-1}(X) \in \mathcal{F}\}.$$

The heart $\mathcal{H}_{\mathfrak{t}}$ of this t -structure consists of the complexes $M \in \mathcal{D}^{[-1,0]}(R)$ such that $H^{-1}(M) \in \mathcal{F}$ and $H^0(M) \in \mathcal{T}$.

Definition 2.1. *Let $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $R\text{-Mod}$. We say that \mathfrak{t} is a t CG torsion pair when $(\mathcal{U}_{\mathfrak{t}}, \mathcal{U}_{\mathfrak{t}}^\perp[1])$ is a compactly generated t -structure.*

Example 2.2. *Let R be a commutative Noetherian ring. Then every hereditary torsion pair $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ in $R\text{-Mod}$ is a t CG torsion pair. Indeed, if $\mathfrak{t} = (\mathcal{T}, \mathcal{F})$ is an hereditary torsion pair, then Proposition 1.1 says that there exists Z an sp-subset of $\text{Spec}(R)$ such that $\mathfrak{t} = (\mathcal{T}, \mathcal{F}) = (\mathcal{T}_Z, \mathcal{T}_Z^\perp)$. Now, we define an sp-filtration as follows:*

$$\phi(n) = \begin{cases} \emptyset & \text{if } n > 0 \\ Z & \text{if } n = 0 \\ \text{Spec}(R) & \text{if } n < 0. \end{cases}$$

By [ATJLS10, Theorem 3.11], we obtain that $\mathcal{U}_\phi = \{X \in \mathcal{D}(R) \mid \text{Supp}(H^j(X)) \subseteq \phi(j), \text{ for all } j \in \mathbb{Z}\} = \mathcal{U}_t$. Now using [ATJLS10, Theorem 3.10], we get that $(\mathcal{U}_\phi, \mathcal{U}_\phi^\perp[1]) = (\mathcal{U}_t, \mathcal{U}_t^\perp[1])$ is a compactly generated t -structure.

The following result characterizes us the t CG torsion pairs in terms of the torsion pair in $R\text{-Mod}$.

Theorem 2.3. *Let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $R\text{-Mod}$. Then, \mathbf{t} is a t CG torsion pair, if and only if, there exists, $\{T_\lambda\}_{\lambda \in \Lambda}$ a set of finitely presented R -modules in \mathcal{T} , such that $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \text{Ker}(\text{Hom}_R(T_\lambda, ?))$.*

Proof. Let $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ be a set of compact generators of the aisle $\mathcal{U}_t = \{X \in D^{\leq 0}(R) : H^0(X) \in \mathcal{T}\}$. Without loss of generality, we can assume that each S_λ is of the form:

$$S_\lambda := \cdots \rightarrow 0 \rightarrow P_\lambda^{n_\lambda} \xrightarrow{d_\lambda^{n_\lambda}} \cdots \rightarrow P_\lambda^0 \xrightarrow{d_\lambda^0} \cdots \xrightarrow{d_\lambda^{m_\lambda-1}} P_\lambda^{m_\lambda} \rightarrow 0 \rightarrow \cdots,$$

where each P_λ^k is a finitely generated projective R -module (see [Ric89]). Since each S_λ is in $D^{\leq 0}(R)$, we obtain that the following exact sequence is split:

$$0 \rightarrow \text{Ker}(d_\lambda^{m_\lambda-1}) = \text{Im}(d_\lambda^{m_\lambda-2}) \rightarrow P_\lambda^{m_\lambda-1} \rightarrow \text{Im}(d_\lambda^{m_\lambda-1}) = P_\lambda^{m_\lambda} \rightarrow 0.$$

Thus $\text{Im}(d_\lambda^{m_\lambda-2})$ is a finitely generated projective R -module. By using this argument in a recursive way, we obtain that $\text{Im}(d_\lambda^0)$ is also a finitely generated projective R -module and thus, so is $\text{Ker}(d_\lambda^0)$. Next, since $\text{Im}(d_\lambda^{-1})$ is a finitely generated R -module, the following exact sequence shows that $H^0(S_\lambda)$ is a finitely presented R -module:

$$0 \rightarrow \text{Im}(d_\lambda^{-1}) \rightarrow \text{Ker}(d_\lambda^0) \rightarrow H^0(S_\lambda) \rightarrow 0.$$

We claim that $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \text{Ker}(\text{Hom}_R(H^0(S_\lambda), ?))$. Indeed, if we fix an R -module M in $\bigcap_{\lambda \in \Lambda} \text{Ker}(\text{Hom}_R(H^0(S_\lambda), ?))$, then it is clear that $t(M)$ also is in $\bigcap_{\lambda \in \Lambda} \text{Ker}(\text{Hom}_R(H^0(S_\lambda), ?))$. Therefore:

$$\begin{aligned} 0 &= \text{Hom}_R(H^0(S_\lambda), t(M)) \\ &\cong \text{Hom}_{\mathcal{D}(R)}(H^0(S_\lambda)[0], t(M)[0]) \\ &\cong \text{Hom}_{\mathcal{D}(R)}(S_\lambda, t(M)[0]) \end{aligned}$$

where the last isomorphism follows by applying the contravariant cohomological functor $\text{Hom}_{\mathcal{D}(R)}(?, t(M)[0])$ to the canonical triangle

$$\tau^{\leq -1}(S_\lambda) \rightarrow S_\lambda \rightarrow H^0(S_\lambda)[0] \xrightarrow{+}.$$

It follows that $\text{Hom}_{\mathcal{D}(R)}(S_\lambda[n], t(M)[0]) = 0$ for all $S_\lambda \in \mathcal{S}$ and integers $n \geq 0$. This implies that $t(M)[0] \in \mathcal{U}_t \cap \mathcal{U}_t^\perp = 0$ and therefore $M \in \mathcal{F}$.

Conversely, for each λ , we will denote by, S_λ , the complex:

$$S_\lambda := \cdots \longrightarrow 0 \longrightarrow R^{(n_\lambda)} \xrightarrow{\quad} R^{(m_\lambda)} \longrightarrow 0 \longrightarrow \cdots$$

$\begin{array}{c} K_\lambda \\ \nearrow \quad \searrow \\ R^{(n_\lambda)} \quad R^{(m_\lambda)} \end{array}$

where m_λ, n_λ are positive integers, $R^{(m_\lambda)}$ is in degree 0, and K_λ is the finitely generated R -module given by the kernel of the epimorphism $R^{(m_\lambda)} \twoheadrightarrow T_\lambda$. Note that $\mathcal{S} = \{S_\lambda\} \cup \{R[1]\}$ is a set of compact complexes in \mathcal{U}_t , and therefore $\text{aisle}(\mathcal{S}) \subseteq \mathcal{U}_t$. On the other hand, let X be a complex in $\text{aisle}(\mathcal{S})^\perp$. Then we obtain that

$$0 = \text{Hom}_{\mathcal{D}(R)}((R[1])[n], X) = \text{Hom}_{\mathcal{D}(R)}(R, X[-1-n]) = H^{-1-n}(X),$$

for all integers $n \geq 0$, showing that $X \in \mathcal{D}^{\geq 0}(R)$.

Now, applying the cohomological functor $\mathrm{Hom}_{\mathcal{D}(R)}(S_\lambda, ?)$ to the triangle

$$H^0(X)[0] \rightarrow X \rightarrow \tau^{>0}(X) \xrightarrow{+},$$

gives that $\mathrm{Hom}_{\mathcal{D}(R)}(S_\lambda, X) \cong \mathrm{Hom}_{\mathcal{D}(R)}(S_\lambda, H^0(X)[0])$. But,

$$\begin{aligned} 0 &= \mathrm{Hom}_{\mathcal{D}(R)}(S_\lambda, X) \\ &\cong \mathrm{Hom}_{\mathcal{D}(R)}(S_\lambda, H^0(X)[0]) \\ &\cong \mathrm{Hom}_{\mathcal{D}(R)}(H^0(S_\lambda)[0], H^0(X)[0]) \\ &= \mathrm{Hom}_R(T_\lambda, H^0(X)). \end{aligned}$$

Since $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \mathrm{Ker}(\mathrm{Hom}_R(T_\lambda, ?))$, it follows that $H^0(X) \in \mathcal{F}$. Therefore, $\mathrm{aisle}(\mathcal{S})^\perp \subseteq \mathcal{U}_t^\perp$, and since $\mathcal{U}_t^\perp \subseteq \mathrm{aisle}(\mathcal{S})^\perp$, then we obtain that $(\mathcal{U}_t, \mathcal{U}_t^\perp[1]) = (\mathrm{aisle}(\mathcal{S}), \mathrm{aisle}(\mathcal{S})^\perp[1])$ is a compactly generated t -structure. \square

The simplest version of Theorem 2.3 is when $\{T_\lambda\}_{\lambda \in \Lambda}$ is formed by one element. In such case $\mathcal{F} = \mathrm{Ker}(\mathrm{Hom}_R(T_\lambda, ?))$. Another way to obtain this condition over \mathcal{F} is when $\mathcal{T} = \mathrm{Gen}(T_\lambda)$, for some finitely presented T_λ ; for example when T_λ is a classical tilting R -module or if \mathcal{H}_t is a module category (see [PS16b, Lemma 3.2]). However, in general, $\mathcal{F} = \mathrm{Ker}(\mathrm{Hom}_R(T_\lambda, ?))$ does not imply $\mathcal{T} = \mathrm{Gen}(T_\lambda)$, in other words, the reciprocal of the following result is not true.

Corollary 2.4. *Let R be a ring. Then, every torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $R\text{-Mod}$ such that $\mathcal{T} = \mathrm{Gen}(V)$, for some finitely presented R -module V , is a $t\mathrm{CG}$ torsion pair.*

Corollary 2.5. *Let R be a ring and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $R\text{-Mod}$. If \mathbf{t} is a $t\mathrm{CG}$ torsion pair, then \mathcal{H}_t is a Grothendieck category.*

Proof. Note that \mathcal{F} is closed under direct limits if \mathbf{t} is a $t\mathrm{CG}$ torsion pair. The result now follows from [PS16a, Theorem 1.2]. \square

Definition 2.6. *A torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ in $R\text{-Mod}$ is said to be of finite type, if $\mathcal{F} = \varinjlim \mathcal{F}$.*

Remark 2.7. *Corollary 2.5 shows that every $t\mathrm{CG}$ torsion pair is of finite type. Hence, using the [PS15, Lemma 4.6], we obtain that the collection of the $t\mathrm{CG}$ torsion pairs actually form a set, that we will denote by $t\mathrm{CG}(R)$.*

The following corollary, relaxes condition 2) in Theorem 2.3 when the torsion pairs is of finite type.

Corollary 2.8. *Let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $R\text{-Mod}$ of finite type. The following assertions are equivalent.*

- (1) \mathbf{t} is a $t\mathrm{CG}$ torsion pair;
- (2) There exists $\{T_\lambda\}_{\lambda \in \Lambda}$, a set of finitely presented R -modules in \mathcal{T} , such that $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \mathrm{Ker}(\mathrm{Hom}_R(T_\lambda, ?))$;
- (3) There exists $\{T_\lambda\}_{\lambda \in \Lambda}$, a set of finitely generated R -modules in \mathcal{T} , such that $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \mathrm{Ker}(\mathrm{Hom}_R(T_\lambda, ?))$.

Proof. By the Theorem 2.3 and since that (2) \Rightarrow (3) is trivial, we just need to prove (3) \Rightarrow (2). This last implication follows directly from the proof of [Hrb16, Lemma 2.4], nevertheless, for clarity, we include some details of this proof. Let $\{T_\lambda\}_{\lambda \in \Lambda}$ be

a set of finitely generated R -modules in \mathcal{T} , such that $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \text{Ker}(\text{Hom}_R(T_\lambda, ?))$. For each λ , we consider the following exact sequence in $R\text{-Mod}$:

$$0 \longrightarrow K \longrightarrow R^{(n_\lambda)} \longrightarrow T_\lambda \longrightarrow 0,$$

where n_λ is a natural number. Now, we fix a direct system $(K_i)_{i \in I}$ of finitely generated submodules of K such that $K = \bigcup_{i \in I} K_i$. Note that for each i , we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_i & \longrightarrow & R^{(n_\lambda)} & \longrightarrow & R^{(n_\lambda)}/K_i \longrightarrow 0 \\ & & \downarrow \iota_i & & \parallel & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & R^{(n_\lambda)} & \longrightarrow & T_\lambda \longrightarrow 0 \end{array}$$

It follows that $T_\lambda \cong \varinjlim R^{(n_\lambda)}/K_i$, where all morphism of this direct system are projections.

Since $\mathcal{F} = \varinjlim \mathcal{F}$, we obtain that $T_\lambda = \varinjlim t(R^{(n_\lambda)}/K_i)$. Next, for each $i \in I$, we consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_i & \longrightarrow & J_i & \longrightarrow & t(R^{(n_\lambda)}/K_i) \rightarrow 0 \\ & & \parallel & & \downarrow \lrcorner & & \downarrow \\ 0 & \longrightarrow & K_i & \longrightarrow & R^{(n_\lambda)} & \longrightarrow & R^{(n_\lambda)}/K_i \rightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & R^{(n_\lambda)}/J_i & \xrightarrow{\sim} & (1 : t)(R^{(n_\lambda)}/J_i) \end{array}$$

Where the top right square of this diagram is the pullback square obtained from $R^{(n_\lambda)} \rightarrow R^{(n_\lambda)}/K_i$ and $t(R^{(n_\lambda)}/K_i) \hookrightarrow R^{(n_\lambda)}/K_i$.

Given that $\varinjlim R^{(n_\lambda)}/J_i \cong \varinjlim (1 : t)(R^{(n_\lambda)}/K_i) = 0$, it follows that $(J_i)_{i \in I}$ is a directed union such that $\bigcup_{i \in I} J_i = R^{(n_\lambda)}$. Since $R^{(n_\lambda)}$ is finitely generated R -module, there is $k \in I$ such that $J_i = R^{(n_\lambda)}$, for all $i \geq k$. Hence $R^{(n_\lambda)}/K_i$ is a finitely presented R -module which is in \mathcal{T} , for all $i \geq k$. If we let $\mathcal{S}_\lambda := \{R^{(n_\lambda)}/K_i : i \geq k\}$, then we obtain that $\mathcal{F} = \bigcap_{S \in \bigcup \mathcal{S}_\lambda} \text{Ker}(\text{Hom}_R(S, ?))$. \square

Theorem 2.9. *Let R be a left Noetherian ring and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $R\text{-Mod}$. The following assertions are equivalent:*

- (1) \mathbf{t} is a *tCG torsion pair*;
- (2) \mathbf{t} is of *finite type*;
- (3) $\mathcal{H}_{\mathbf{t}}$ is a *Grothendieck category*.

Proof. By [PS16a, Theorem 1.2] and Corollary 2.5, we just need to prove (2) \Rightarrow (1). From [PS15, Lemma 4.6], we can assume that $\mathcal{T} = \text{Pres}(V) = \text{Gen}(V)$, for some R -module V . Now, we fix a direct system $(V_\lambda)_{\lambda \in \Lambda}$ of finitely generated submodules of V , such that $\varinjlim V_\lambda = V$. By hypothesis, $\mathcal{F} = \varinjlim \mathcal{F}$, then we get that $\varinjlim t(V_\lambda) \cong V$. Furthermore, each $t(V_\lambda)$ is a finitely generated R -module, since R is a left Noetherian ring. Note that $\mathcal{F} = \text{Ker}(\text{Hom}_R(V, ?)) = \text{Ker}(\text{Hom}_R(\varinjlim t(V_\lambda), ?))$, therefore $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \text{Ker}(\text{Hom}_R(t(V_\lambda), ?))$, and hence the result follows from Corollary 2.8. \square

The next result is an immediate corollary of Theorem 2.9 and [ATJLS10, Theorem 3.10]. However, this result is also given in [AH16, Lemma 4.2], but from a different point of view, namely the theory of *silting modules*.

Corollary 2.10. *Let R be a commutative Noetherian ring and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $R\text{-Mod}$. Then $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is an hereditary torsion pair if, and only if, \mathcal{F} is closed under direct limits.*

Proof. If $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is an hereditary torsion pair in $R\text{-Mod}$, then we have that $\mathcal{F} = \bigcap_{\mathbf{a} \in F_g} \text{Ker}(\text{Hom}_R(R/\mathbf{a}, ?))$, where F_g is the Gabriel filter associated to the torsion pair \mathbf{t} (see [Ste75]). Since R is Noetherian ring, then R/\mathbf{a} is finitely presented as R -module and so \mathcal{F} is closed under direct limits.

Conversely, from Theorem 2.9, we get that $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is a $t\text{CG}$ torsion pair. But, by [ATJLS10, Theorem 3.10], there exists an sp-filtration $\phi : \mathbb{Z} \rightarrow \text{P}(\text{Spec}(R))$, such that $\mathcal{U}_{\mathbf{t}} = \mathcal{U}_{\phi}$. From the description of \mathcal{U}_{ϕ} , we see that $\phi(k) = \emptyset$, for all integers $k > 0$ and $\phi(k) = \text{Spec}(R)$, for all integers $k \leq -1$. Moreover, $\mathcal{T} = \mathcal{T}_0$, where $(\mathcal{T}_0, \mathcal{F}_0)$ is the hereditary torsion pair associated to the sp-subset $\phi(0)$. \square

Recall that a ring R is called *left coherent ring* if each finitely generated left ideal of R is a finitely presented R -module. It is well known fact that if R is a left coherent ring, then the class $fp(R\text{-Mod})$, of finitely presented R -modules, is an abelian category. The class of all the torsion pairs in $fp(R\text{-Mod})$, will be denoted by $\mathbf{t}(fp(R))$.

The following result shows us a way to get $t\text{CG}$ torsion pair, for left coherent rings. However there are $t\text{CG}$ torsion pairs that cannot be obtained in this way.

Theorem 2.11. *Let R be a left coherent ring. The assignment $\mathbf{t} = (\mathcal{X}, \mathcal{Y}) \rightsquigarrow \tilde{\mathbf{t}} = (\varinjlim \mathcal{X}, \varinjlim \mathcal{Y})$ defines an injective function $\phi : \mathbf{t}(fp(R)) \rightarrow t\text{CG}(R)$.*

Moreover, the following assertions hold:

- (1) *If R is a Noetherian ring, then ϕ is a bijective function.*
- (2) *Let $R := \prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ with addition and multiplication defined component wise. For this coherent ring, ϕ is not bijective.*

Proof. Let $\mathbf{t} = (\mathcal{X}, \mathcal{Y})$ be a torsion pair in $fp(R\text{-Mod})$. By [CB94, Lemma 4.4], we know that the torsion pair in $R\text{-Mod}$ generated by \mathcal{X} is $\tilde{\mathbf{t}} = (\mathcal{T}, \mathcal{F}) = (\varinjlim \mathcal{X}, \varinjlim \mathcal{Y})$. Moreover, $\mathcal{F} = \varinjlim \mathcal{Y}$ consists of the R -modules F such that $\text{Hom}_R(X, F) = 0$, for all $X \in \mathcal{X}$. Hence \mathcal{F} is closed under direct limits, and therefore there exist V an R -module such that $\mathcal{T} = \varinjlim \mathcal{X} = \text{Gen}(V)$, see [PS15, Lemma 4.6]. Now, we take a direct system $(X_{\lambda})_{\lambda \in \Lambda}$ in \mathcal{X} such that $V = \varinjlim X_{\lambda}$. Since, $\mathcal{F} = \text{Ker}(\text{Hom}_R(V, ?)) = \text{Ker}(\text{Hom}_R(\varinjlim X_{\lambda}, ?))$, we get that $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \text{Ker}(\text{Hom}_R(X_{\lambda}, ?))$. Thus, by Theorem 2.3 we obtain that $\tilde{\mathbf{t}} = (\mathcal{T}, \mathcal{F}) = (\varinjlim \mathcal{X}, \varinjlim \mathcal{Y})$ is a $t\text{CG}$ torsion pair.

On the other hand, if $\mathbf{t}_1 = (\mathcal{X}_1, \mathcal{Y}_1)$ is a torsion pair in $fp(R\text{-Mod})$ such that $\tilde{\mathbf{t}}_1 = (\varinjlim \mathcal{X}_1, \varinjlim \mathcal{Y}_1) = (\varinjlim \mathcal{X}, \varinjlim \mathcal{Y}) = \tilde{\mathbf{t}}$. Then, for each $X_1 \in \mathcal{X}_1$, we have that $(1 : t)(X_1) \in \mathcal{X}_1 \cap \mathcal{Y} \subseteq \varinjlim \mathcal{X}_1 \cap \varinjlim \mathcal{Y} = \varinjlim \mathcal{X} \cap \varinjlim \mathcal{Y} = 0$, hence $X_1 \in \mathcal{X}$ and therefore $\mathcal{X}_1 \subseteq \mathcal{X}$. By a similar argument we obtain that $\mathcal{X} \subseteq \mathcal{X}_1$, so that $\mathbf{t} = \mathbf{t}_1$. This show that the assignment $\mathbf{t} \rightsquigarrow \tilde{\mathbf{t}}$ is one-to-one, and since $t\text{CG}(R)$ is a set, it follows that $\mathbf{t}(fp(R))$ is also a set. Hence the assignment $\mathbf{t} \rightsquigarrow \tilde{\mathbf{t}}$ is a function that is injective. This complete the first part of the theorem.

We now suppose that R is a Noetherian ring. Let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a $t\text{CG}$ torsion pair in $R\text{-Mod}$. We will show that there is $\mathbf{t}_1 \in \mathbf{t}(fp(R))$, such that $\phi(\mathbf{t}_1) = \mathbf{t}$.

Now, if $P \in fp(R\text{-Mod})$, then we have the following exact sequence in $R\text{-Mod}$:

$$0 \longrightarrow t(P) \longrightarrow P \longrightarrow (1 : t)(P) \longrightarrow 0.$$

Note that $t(P)$ is a finitely generated R -module, which in this case is also a finitely presented R -module. Therefore, the previous exact sequence is also in $fp(R\text{-Mod})$. This shows that $\mathbf{t}_1 := (\mathcal{T} \cap fp(R\text{-Mod}), \mathcal{F} \cap fp(R\text{-Mod}))$ is a torsion pair in $fp(R\text{-Mod})$. Since \mathcal{F} is closed under direct limits, we obtain that $\varinjlim(\mathcal{T} \cap fp(R\text{-Mod})) \subseteq \mathcal{T}$ and that $\varinjlim(\mathcal{F} \cap fp(R\text{-Mod})) \subseteq \mathcal{F}$, and hence $\phi(\mathbf{t}_1) = \mathbf{t}$.

For the final statement of the theorem, let $R := \prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ and let $\mathbf{a} := \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. It is clear that \mathbf{a} is an idempotent two-sided ideal of R , hence by [Ste75, Proposition VI.6.12] we consider the TTF triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ in $R\text{-Mod}$ associated to the ideal \mathbf{a} . In other words, $\mathbf{t}_1 = (\mathcal{C}, \mathcal{T})$ and $\mathbf{t}_2 = (\mathcal{T}, \mathcal{F})$ are torsion pair in $R\text{-Mod}$, and we have that $\mathcal{C} = \text{Gen}(\mathbf{a}) = \{C \in R\text{-Mod} : \mathbf{a}C = C\}$, $\mathcal{T} = \{T \in R\text{-Mod} : \mathbf{a}T = 0\}$ and $\mathcal{F} = \text{Ker}(\text{Hom}_R(R/\mathbf{a}, ?))$.

Next, for each $n \in \mathbb{N}$, we consider $\bigoplus_{j=1}^n \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota_n} \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$, the canonical inclusion, and let $I_n := \text{Im}(\iota_n)$. Note that \mathbf{a} is the direct union of the I_n , and that every I_n is a finitely generated R -module such that $\mathbf{a}I_n = I_n$.

Since $\mathcal{T} = \text{Ker}(\text{Hom}_R(\mathbf{a}, ?)) = \text{Ker}(\text{Hom}_R(\bigcup_{n \in \mathbb{N}} I_n, ?))$ and that each $I_n \in \mathcal{C}$, it follows that $\mathcal{T} = \bigcap_{n \in \mathbb{N}} \text{Ker}(\text{Hom}_R(I_n, ?))$. On the other hand, note that the torsion pair \mathbf{t}_1 is of finite type, then by Corollary 2.8, we obtain that $\mathbf{t}_1 = (\mathcal{C}, \mathcal{T})$ is a $t\text{CG}$ torsion pair.

We claim that $\mathbf{t}_1 \notin \text{Im}(\phi)$. Indeed, suppose that there exists $\mathbf{t} = (\mathcal{X}, \mathcal{Y})$ torsion pair in $fp(R\text{-Mod})$ such that $\phi(\mathbf{t}) = \mathbf{t}_1$. Now, we consider the following exact sequence in $fp(R\text{-Mod})$:

$$0 \longrightarrow t(R) \longrightarrow R \longrightarrow (1 : t)(R) \longrightarrow 0.$$

Since $t(R) \in \mathcal{X} \subseteq \varinjlim \mathcal{X} = \mathcal{C}$ and that $(1 : t)(R) \in \mathcal{Y} \subseteq \varinjlim \mathcal{Y} = \mathcal{T}$, we obtain that $t(R) = \mathbf{a}$ and that $(1 : t)(R) = \frac{R}{\mathbf{a}}$, but \mathbf{a} is not finitely generated. This is a contradiction. \square

3. CONSTRUCTION OF $t\text{CG}$ TORSION PAIRS AND TORSION PAIRS OF FINITE TYPE

In the previous section, we showed that every $t\text{CG}$ torsion pair is of finite type. Therefore, it is natural to ask when the converse is true or what separates us from this fact.

Corollary 3.1. *Let R be a ring and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be an hereditary torsion pair in $R\text{-Mod}$. Then, \mathbf{t} is a $t\text{CG}$ torsion pair if, and only if, \mathbf{t} is of finite type.*

Proof. Suppose that \mathbf{t} is of finite type. From [Hrb16, Lemma 2.4] and its proof, there exists $\{T_\lambda\}_{\lambda \in \Lambda}$ a set of finitely presented R -modules in \mathcal{T} such that $\mathcal{F} = \{M \in R\text{-Mod} \mid \text{Hom}_R(T_\lambda, M) = 0, \text{ for all } \lambda \in \Lambda\}$. The result now follows from Theorem 2.3. \square

Proposition 3.2. *Let R be a ring and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $R\text{-Mod}$. If $\mathcal{T} = \text{Gen}(V)$ for some R -module V , in particular if \mathbf{t} is of finite type, then there exists $\tilde{\mathbf{t}} = (\tilde{\mathcal{T}}, \tilde{\mathcal{F}})$ a $t\text{CG}$ torsion pair, such that $\mathcal{T} \subseteq \tilde{\mathcal{T}}$.*

Proof. Let (V_λ) be a direct system of finitely presented R -modules so that $\varinjlim V_\lambda = V$. For each λ , we will denote by S_λ , the complex:

$$S_\lambda := \cdots \longrightarrow 0 \longrightarrow R^{(n_\lambda)} \xrightarrow{d_\lambda} R^{(m_\lambda)} \longrightarrow 0 \longrightarrow \cdots$$

$\begin{array}{c} K_\lambda \\ \nearrow \quad \searrow \\ R^{(n_\lambda)} \quad R^{(m_\lambda)} \end{array}$

where m_λ, n_λ are positive integers, $R^{(m_\lambda)}$ is in degree 0, and K_λ is the finitely generated R -module given by the kernel of the epimorphism $R^{(m_\lambda)} \twoheadrightarrow V_\lambda$. Note that $\mathcal{S} = \{S_\lambda\} \cup \{R[1]\}$ is a set of compact complexes in $\mathcal{D}(R)$, and therefore $(\text{aisle}(\mathcal{S}), \text{aisle}(\mathcal{S})^\perp[1])$ is a compactly generated t -structure; its heart will be denoted by $\mathcal{H}_\mathcal{S}$.

Since $\mathcal{S} \subseteq \mathcal{D}^{\leq 0}(R)$, then $\text{aisle}(\mathcal{S}) \subseteq \mathcal{D}^{\leq 0}(R)$. On the other hand, the fact of that $R[1] \in \mathcal{S}$ implies that $\text{aisle}(\mathcal{S})^\perp \subseteq \mathcal{D}^{\geq 0}(R)$, thus $\mathcal{H}_\mathcal{S} \subseteq \mathcal{D}^{[-1,0]}(R)$. We claim that $\tilde{\mathcal{T}} := \{X \in R\text{-Mod} \mid \exists M \in \mathcal{H}_\mathcal{S} \text{ with } H^0(M) = X\}$ is a torsion class that contains \mathcal{T} , and the corresponding torsion pair $\tilde{\mathfrak{t}} := (\tilde{\mathcal{T}}, \tilde{\mathcal{T}}^\perp)$, satisfies that $\mathcal{H}_{\tilde{\mathfrak{t}}} = \mathcal{H}_\mathcal{S}$. Indeed, note that $\text{aisle}(\mathcal{S})^\perp$ is closed under coproducts and since the cohomological functor $H^0 : \mathcal{D}(R) \rightarrow R\text{-Mod}$ commutes with coproducts, we obtain that $\tilde{\mathcal{T}}$ is closed under coproducts.

Let M be a complex in $\mathcal{H}_\mathcal{S}$. Since $\text{Hom}_{\mathcal{D}(R)}(M, X) = 0$, for all $X \in \text{aisle}(\mathcal{S})^\perp \subseteq \mathcal{D}^{\geq 0}(R)$ and $M \in \mathcal{D}^{[-1,0]}(R)$, then we obtain that $\text{Hom}_R(H^0(M), H^0(X)) = 0$, for all $X \in \text{aisle}(\mathcal{S})^\perp$. It's not difficult to see that $\tilde{T}[0] \in \text{aisle}(\mathcal{S})$ (moreover, $\tilde{T}[0] \in \mathcal{H}_\mathcal{S}$) if, and only if, $\text{Hom}_R(\tilde{T}, H^0(X)) = 0$, for all $X \in \text{aisle}(\mathcal{S})^\perp$.

Now let $\tilde{T} \in \tilde{\mathcal{T}}$, then there is $M \in \mathcal{H}_\mathcal{S}$ such that $H^0(M) = \tilde{T}$. From the first part of the previous paragraph, we know that $\text{Hom}_R(\tilde{T}, H^0(X)) = 0$, for all $X \in \text{aisle}(\mathcal{S})^\perp$. Thus, $\tilde{\mathcal{T}} \subseteq \{\tilde{T} \in R\text{-Mod} \mid \text{Hom}_R(\tilde{T}, H^0(X)) = 0, \text{ for all } X \in \text{aisle}(\mathcal{S})^\perp\}$, and the other inclusion follows from the last claim of the previous paragraph. Therefore, $\tilde{\mathcal{T}} = \{\tilde{T} \in R\text{-Mod} \mid \text{Hom}_R(\tilde{T}, H^0(X)) = 0, \text{ for all } X \in \text{aisle}(\mathcal{S})^\perp\}$, and one sees that $\tilde{\mathcal{T}}$ is closed under quotients and extensions, that is, $\tilde{\mathcal{T}}$ is a torsion class. Finally, note that each $V_\lambda \in \tilde{\mathcal{T}}$, and hence $V \in \tilde{\mathcal{T}}$, but by hypothesis $\mathcal{T} = \text{Gen}(V)$, so that $\mathcal{T} \subseteq \tilde{\mathcal{T}}$.

The previous argument, shows that $\mathcal{H}_\mathcal{S} \subseteq \mathcal{H}_{\tilde{\mathfrak{t}}}$ and $\tilde{\mathcal{T}}[0] \subseteq \mathcal{H}_\mathcal{S}$. Let F be a R -module in $\tilde{\mathcal{T}}^\perp$, and note that $F[1] \in \text{aisle}(\mathcal{S})^\perp[1]$, since each V_λ is in $\tilde{\mathcal{T}}$. It's easy to see that $\text{Hom}_{\mathcal{D}(R)}(F[1], X) = 0$, for all $X \in \text{aisle}(\mathcal{S})^\perp \subseteq \mathcal{D}^{\geq 0}(R)$, this implies that $\tilde{\mathcal{T}}^\perp[1] \subseteq \mathcal{H}_\mathcal{S}$ and, hence, $\mathcal{H}_\mathcal{S} = \mathcal{H}_{\tilde{\mathfrak{t}}}$.

Lastly to see that $\tilde{\mathfrak{t}}$ is a t CG torsion pair, note that

$$\begin{aligned} \mathcal{U}_{\tilde{\mathfrak{t}}} &= \{M \in \mathcal{D}^{\leq 0}(R) : H^0(M) \in \tilde{\mathcal{T}}\} \\ &= \{M \in \mathcal{D}^{\leq 0}(R) : \text{Hom}_R(H^0(M), H^0(X)) = 0, \text{ for all } X \in \text{aisle}(\mathcal{S})^\perp\} \\ &= \{M \in \mathcal{D}^{\leq 0}(R) : \text{Hom}_{\mathcal{D}(R)}(H^0(M)[0], X) = 0, \text{ for all } X \in \text{aisle}(\mathcal{S})^\perp\} \\ &= \{M \in \mathcal{D}^{\leq 0}(R) : \text{Hom}_{\mathcal{D}(R)}(M, X) = 0, \text{ for all } X \in \text{aisle}(\mathcal{S})^\perp\} \\ &= \text{aisle}(\mathcal{S}) \end{aligned}$$

This shows that $\tilde{\mathfrak{t}} = (\tilde{\mathcal{T}}, \tilde{\mathcal{T}}^\perp)$ is a t CG torsion pair. \square

A ring R is called *von Neumann regular ring*, if for every a in R , there exists x in R such that $a = axa$. It is well known fact that if R is a von Neumann regular ring, then every finitely presented R -module is a projective R -module.

Lemma 3.3. *Let R be a von Neumann regular ring. If \mathbf{a} is a two-sided ideal of R , then for each $a \in \mathbf{a}$, we get $\mathbf{a}(Ra) = Ra$.*

Proof. Note that $\mathbf{a}(Ra) = (\mathbf{a}R)a = \mathbf{a}a$. On other hand, there exists x in R , such that $axa = a$. Thus, for every $r \in R$, we get $ra = (rax)a$ and since \mathbf{a} is a two-sided ideal of R , we obtain that $ra \in \mathbf{a}a$, so that $Ra \subseteq \mathbf{a}a$. This shows that $Ra \subseteq \mathbf{a}a = \mathbf{a}(Ra) \subseteq Ra$. \square

The following result gives us an explicit description of tCG torsion pair over a von Neumann regular ring.

Theorem 3.4. *Let R be a von Neumann regular ring and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $R\text{-Mod}$. The following assertions are equivalent:*

- (1) \mathbf{t} is a tCG torsion pair;
- (2) There exists $\{T_\lambda\}_{\lambda \in \Lambda}$, a set of finitely generated projective R -modules such that $\mathcal{T} = \text{Gen}(\coprod T_\lambda)$;
- (3) There exists an unique idempotent two-sided ideal \mathbf{a} of R such that $\mathcal{T} = \text{Gen}(\mathbf{a}) = \{T \in R\text{-Mod} : \mathbf{a}T = T\}$;
- (4) \mathbf{t} is the left constituent pair of a TTF triple in $R\text{-Mod}$.

Proof. (1) \Rightarrow (2). By Theorem 2.3, there exists $\{T_\lambda\}_{\lambda \in \Lambda}$, a set of finitely presented R -modules in \mathcal{T} , such that $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \text{Ker}(\text{Hom}_R(T_\lambda, ?))$. Since R is a von Neumann regular, we get each T_λ is a finitely generated projective R -module. Therefore, $\coprod T_\lambda$ is also a projective R -module and hence $\text{Gen}(\coprod T_\lambda)$ is a torsion class in $R\text{-Mod}$. In this case, the equality $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \text{Ker}(\text{Hom}_R(T_\lambda, ?))$, implies $\mathcal{T} = \text{Gen}(\coprod T_\lambda)$.

(2) \Rightarrow (3). It follows from [Ste75, Proposition VI.9.4] and [Ste75, Corollary VI.9.5].

(3) \Rightarrow (4). It follows from [Ste75, Chapter VI, Section 8].

(4) \Rightarrow (1). We suppose that there exists a TTF triple in $R\text{-Mod}$ of the form $(\mathcal{T}, \mathcal{F}, \mathcal{F}^\perp)$. Hence, $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is a torsion pair of finite type, and there exists an unique idempotent two-sided ideal \mathbf{a} of R , such that $\mathcal{T} = \text{Gen}(\mathbf{a}) = \{T \in R\text{-Mod} : \mathbf{a}T = T\}$. Now, we fix a direct system $(I_i)_{i \in I}$ of finitely generated submodules of \mathbf{a} such that $\mathbf{a} = \bigcup_{i \in I} I_i$. By Lemma 3.3, we have $\mathbf{a}I_i = I_i$, for all $i \in I$, hence each I_i is in \mathcal{T} . The result follows from Corollary 2.8, since $\mathcal{F} = \text{Ker}(\text{Hom}_R(\mathbf{a}, ?)) = \text{Ker}(\text{Hom}_R(\bigcup_{i \in I} I_i, ?)) = \bigcap_{i \in I} \text{Ker}(\text{Hom}_R(I_i, ?))$. \square

Combining Proposition 3.2 and Theorem 3.4 we get that, over von Neumann regular rings, the torsion class of any torsion pair of finite type is contained in the torsion class of the left constituent pair of a TTF triple in $R\text{-Mod}$. However we don't know if every torsion pair of finite type is the left constituent pair of a TTF triple in $R\text{-Mod}$. For this reason we finalize this article with the following question.

Question 3.5. *Let R be a ring and $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair such that \mathcal{F} is closed under direct limits. Is \mathbf{t} a tCG torsion pair?*

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